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A Reynolds stress model of turbulence and its application to thin shear flows

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The paper provides a model of turbulence which effects closure through approximated transport equations for the Reynolds stress tensor $\overline{u_i u_j}$ and for the turbulence energy-dissipation rate ϵ. In its most general form the model thus entails the solution of seven transport equations for turbulence quantities but contains only six constants to be determined by experiment. It is demonstrated that the proposed approximation to the pressure–rate-of-strain correlations leads to satisfactory predictions of the component stress levels in plane homogeneous turbulence, including the non-equality of the lateral and transverse normal-stress components.

For boundary-layer flows a simpler version of the model is derived wherein transport equations are solved only for the shear stress $-\overline{u_1 u_2}$, the turbulence energy k, and ϵ. This model has been incorporated in the numerical solution procedure of Patankar & Spalding (1970) and applied to the prediction of a number of boundary-layer flows including examples of flow remote from walls, those developing along one wall and those confined within ducts. Three of the flows are strongly asymmetric with respect to the surface of zero shear stress and here the turbulent shear stress does not vanish where the mean rate of strain goes to zero. In most cases the predicted profiles and other quantities accord with the data within the probable accuracy of the measurements.

1. Introduction

Over the past few years a number of general and economical numerical procedures have been developed for solving the systems of strongly nonlinear partial differential equations which describe the dynamic behaviour of a viscous fluid. The arrival of these procedures has had (and continues to have) a two-pronged influence on research in turbulent flows. First, it has shifted the emphasis of research directly towards prescribing the Reynolds stresses; for the numerical procedures in question solve the time-averaged equations of motion which contain the Reynolds stresses as unknowns. Thus, such matters as parametric descriptions of mean-velocity profiles, entrainment formulae or overall mean-flow energy-dissipation rates (all of which, in the past fifteen years or so, have appeared as important components of one or more integral procedure for turbulent boundary layers) have taken on a consequential, rather than causative, role in

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the evolution of the flow. Second, the very fact of being able to solve systems of coupled partial differential equations has stimulated the development and use of more elaborate closing approximations for the Reynolds stresses than had formerly been employed.

Bradshaw, Ferriss & Atwell (1967) provided what was perhaps the first widely used boundary-layer prediction procedure in which a differential transport equation was used to determine the shear stress, the dependent variable in question being the turbulence energy. Nee & Kovaznay (1968) have also proposed a turbulence model which entails the solution of one differential equation; they employed the Boussinesq 'effective-viscosity' concept with the effective viscosity itself appearing as the dependent variable of a differential transport equation.

A limitation of these one-equation models arises from the fact that the turbulence length scale (appropriate to the energy-containing motions) which appears in the respective turbulence transport equations is prescribed as an *algebraic* function of position in the boundary layer. In practice, the proposed length-scale functions are not widely applicable; they are not appropriate to the majority of flows arising outside the laboratory. Indeed, in many circumstances, if one neglects the effects of convection and diffusion on the length scale one might just as well stick to Prandtl's (1925) mixing-length hypothesis.

A number of works have sought to provide models of wider applicability by supplying a transport equation from which the length scale may be determined; here may be mentioned, for example, the work of Harlow & Nakayama (1968), Rodi & Spalding (1970), Ng & Spalding (1969), Spalding (1970) and Jones & Launder (1972). Each of these models provides an equation for the turbulent kinetic energy in addition to a scale-determining equation. Closure is thus accomplished through the Prandtl–Kolmogorov formula for the effective turbulent viscosity ν_T:

$$\nu_T = k^{\frac{1}{2}}l,$$

where k denotes the turbulence kinetic energy and l a length scale proportional to that of the energy-containing motions. Since an equation for the turbulence energy is solved, it is clearly not essential for the dependent variable of the second transport equation to be the length scale itself; any variable of the form $k^a l^b$ would be suitable. Thus Ng & Spalding and Rodi & Spalding have used an equation for the energy–length-scale product while Harlow & Nakayama and Jones & Launder have preferred the energy dissipation rate, which at high turbulence Reynolds numbers may be interpreted as $k^{\frac{3}{2}}/l$.

In a number of instances turbulence models of the two-equation type have brought accord between experiment and prediction where hitherto there had been none. For example, the model of Rodi & Spalding correctly predicts the rate of spread of the plane mixing layer, the plane jet and the radial jet. With the Prandtl mixing-length model, however, the mixing length must be taken as 7 %, 9 % and 13 % of the width of the respective flows in order that the growth rate of the shear flows be in agreement with experiment. Likewise, predictions of Jones & Launder (1972) have indicated that in a severe acceleration the length scale in the vicinity of a wall is diminished, causing the mean-flow properties of the boundary layer to display features more akin to those of a laminar than those of a turbulent flow. Again, the results are in quantitative agreement with the available data.

At the time of writing, the predictive capabilities of two-equation representations of turbulent motion are far from fully exploited. It appears, however, that there are many kinds of turbulent flow whose satisfactory description will require a higher order closure of the Reynolds equations than is implied by the effective-viscosity concept. The circumstances in question are those where transport effects upon the Reynolds stresses are appreciable. By way of example, in a wall jet or in flow through an annulus, diffusive transport of shear stress results in the non-coincidence of the surfaces of maximum velocity and zero shear stress, a phenomenon which cannot elegantly be incorporated in the notion of an effective viscosity.

It is the purpose of the present paper to describe the nature and the performance of a model of turbulence which, because it provides transport equations for the Reynolds stresses themselves, does not suffer from the limitations of effective-viscosity models. The proposed model traces its parentage to the early work of Rotta (1951) (and to the even earlier work of Chou (1945)). In its most general form the model provides transport equations for all the Reynolds stresses but, for the present, experimental comparison has been restricted to two-dimensional boundary-layer flows wherein the influence of the normal stresses is acknowledged to be small. The model has therefore been simplified to one where the turbulent shear stress and the turbulence energy (i.e. half the sum of the normal stresses) are the only turbulent velocity correlations determined from transport equations. In accord with the practice of Jones & Launder (1972), the length scale of turbulence is obtained from the solution of an equation for the kinematic energy-dissipation rate. This three-equation model of turbulence, which is developed in §§2–4, contains only six empirical constants of which two may be determined from the observed decay of turbulence in the absence of mean strain.

In §5 a detailed comparison of predictions generated by the model is made with six substantially different nearly parallel flows, including free shear flows, external wall boundary layers and flows within ducts. With few exceptions, the predicted profiles of mean and turbulence quantities agree with the measurements within the probable accuracy of the data.

2. The Reynolds stress equations

For a fluid of uniform density ρ the exact transport equations for the Reynolds stresses $\overline{u_i u_j}$ may be expressed in the form

$$\underbrace{\frac{D\overline{u_i u_j}}{Dt}}_{\text{Convection}} = -\underbrace{\left[\overline{u_j u_k}\frac{\partial U_i}{\partial x_k} + \overline{u_i u_k}\frac{\partial U_j}{\partial x_k}\right]}_{\text{Generation}} - \underbrace{\left[2\nu\overline{\left(\frac{\partial u_i}{\partial x_k}\right)\left(\frac{\partial u_j}{\partial x_k}\right)}\right]}_{\text{Destruction}} + \underbrace{\left[\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}\right]}_{\text{'Redistribution'}}$$

$$-\underbrace{\left\{\frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k} - \nu\frac{\partial \overline{u_i u_j}}{\partial x_k} + \overline{\frac{p}{\rho}(\delta_{jk}u_i + \delta_{ik}u_j)}\right]\right\}}_{\text{Diffusion}}. \quad (2.1)$$

Equation (2.1), in common with the remainder of the paper, adopts tensor notation with repeated suffices indicating summation. Lower and upper case u's denote fluctuating and time averaged velocity components respectively, p denotes fluctuating pressures and overbars imply the usual time averaging of the correlations in question.

In its present form of course, the Reynolds stress equation is not immediately employable in a model of turbulent motion; the right-hand side of (2.1) contains a number of correlations of turbulence quantities for whose determination a closed path must first be prescribed. Indeed, turbulence models may conveniently be categorized by reference to their treatment of (2.1). Thus the neglect of the convective and diffusive transport terms in (2.1) and the algebraic approximation of the remaining ones leads, under favourable circumstances, to the constitutive relation between the Reynolds stress and mean rate of strain employed by effective-viscosity models. If, alternatively, the transport terms are retained but, as above, all the unknown correlations are approximated by expressions containing mean velocity gradients, Reynolds stresses and length scales alone, then the level of closure is of the kind adopted by Rotta (1951), Harlow & Hirt (1969) and Donaldson (1968). Still higher order closures of the Reynolds stress equation have been proposed by Chou (1945), Davidov (1961) and Kolavandin & Vatutin (1969) and entail the provision of a set of transport equations for the triple velocity correlations $\overline{u_i u_j u_k}$ and, for the last of the above, for the microscales of turbulence pertaining to the dissipation terms as well.

The closing approximations which we adopt below place our model within the same category as Rotta's. We eschewed an effective-viscosity model because it was especially our intent to account for transport effects on the stresses. On the other hand, it seemed inappropriate to adopt as elaborate a treatment of the triple correlations as Chou's, for example, when these terms are normally of minor importance compared with the 'redistribution' terms in (2.1) for which only comparatively primitive simulations have been devised.

The following paragraphs describe the restrictions accepted and the assumptions made in order to simplify the stress equations to a practically useful form. The major limitation is that the model should be applicable only to those flow regions where the local turbulence Reynolds number is high. Under this condition, it may be presumed that the smallest scales of motion (which are predominantly responsible for the correlation $\overline{(\partial u_i/\partial x_k)(\partial u_j/\partial x_k)}$) are isotropic. Consequently one replaces the dissipation term in (2.1) by

$$2\nu\overline{\left(\frac{\partial u_i}{\partial x_k}\right)\left(\frac{\partial u_j}{\partial x_k}\right)} = \tfrac{2}{3}\delta_{ij}\epsilon. \tag{2.2}$$

The requirement of high Reynolds number also enables the viscous diffusion term in (2.1) to be dropped. The two remaining diffusion terms, however, cannot be dealt with so certainly. Let us consider the pressure diffusion term first. A companion experimental study of flow in an asymmetric plane channel (Hanjalić & Launder 1972) suggests that $d\overline{pu_2}/dx_2$ is small compared with the other terms appearing in the conservation equation for turbulence energy (x_2 being the coordinate direction normal to the planes). It may not be correct to take this single result as generally indicative of the unimportance of the pressure diffusion terms, nevertheless, in the absence of any other firm evidence, this is the assumption which is made; the term is accordingly neglected.

We stated above our opinion that the provision of transport equations for the triple velocity correlations represented an inappropriate level of closure. It is, none the less, to the equations for $\overline{u_i u_j u_k}$ that we turn to arrive at an appropriate *algebraic* simulation of the term. In appendix A it is shown that with certain assumptions the triple correlations may be replaced by the following form containing only second-order correlations:

$$-\overline{u_i u_j u_k} = c_s \frac{k}{\epsilon}\left[\overline{u_i u_l}\frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l}\frac{\partial \overline{u_k u_i}}{\partial x_l} + \overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\right], \tag{2.3}$$

where c_s is a constant. The multiplier of (2.3) (k/ϵ) may be interpreted as a time scale characteristic of the energy containing and (as is demonstrated in appendix B) of the diffusing motions.

To complete the simulation of equation (2.1), attention is now given to the pressure rate-of-strain correlation

$$\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}.$$

This term is commonly referred to as a 'redistribution' term for, in the equations for the normal stresses (i.e. $i = j$), it is readily demonstrated that the term acts so as to diminish the difference between the normal-stress components (Hinze 1959). Following Chou, the exact expression† for the pressure rate-of-strain correlation at some point $\mathbf{x}_0$ may be written as

$$\begin{aligned}\overline{\frac{p}{\rho}\frac{\partial u_i}{\partial x_j}} &= \frac{1}{4\pi}\int_{\text{vol}}\left[\overline{\left(\frac{\partial^2 u_l u_m}{\partial x_l \partial x_m}\right)'\left(\frac{\partial u_i}{\partial x_j}\right)} + 2\left(\frac{\partial U_l}{\partial x_m}\right)'\overline{\left(\frac{\partial u_m}{\partial x_l}\right)'\left(\frac{\partial u_i}{\partial x_j}\right)}\right]\frac{d\,\text{vol}}{\mathbf{x}} \\ &\equiv \phi_{ij,1} + \phi_{ij,2}.\end{aligned} \tag{2.4}$$

In the above expression terms without superscripts are evaluated at $\mathbf{x}_0$, while those with a prime superscript are evaluated at $(\mathbf{x}_0 + \mathbf{x})$.

Equation (2.4) shows that the correlation in question originates from two types of physical process. The first part is generated from a mutual interaction between turbulence components, while the second arises from the mean rate of strain and its interaction with the turbulence.

In a non-isotropic homogeneous flow with small or zero mean rate of strain only the first part of the pressure-strain term is significant. Since such a flow will decay to the statistically more probable isotropic state, the process denoted above by $\phi_{ij,1}$ must proceed in such a way as to equalize the normal-stress components and to diminish to zero the shear stresses. Such reasoning led Rotta (1951) to propose the following plausible form for the term:

$$(\phi_{ij} + \phi_{ji})_1 = -c_{\phi 1}(\epsilon/k)\,(\overline{u_i u_j} - \tfrac{1}{3}\delta_{ij}2k), \tag{2.5}$$

† Away from the immediate vicinity of a wall.

where $c_{\phi 1}$ is a constant. Later, Rotta (1962) showed that Uberoi's (1957) data provided support for the above approximation, at least for $i = j$. Equation (2.5) is the form adopted here.

Considering next the second part of the pressure-strain term, it is noted that if the turbulence were homogeneous (with the mean velocity varying linearly at right angles to its vector direction) the term $\phi_{ij,2}$ could strictly be re-expressed as

$$\phi_{ij,2} = (\partial U_l/\partial x_m)\, a_{lj}^{mi}, \tag{2.6}$$

where
$$a_{lj}^{mi} \equiv -\frac{1}{2\pi}\int_{\text{vol}} \frac{\partial^2 \overline{u'_m u_i}}{\partial \xi_l \partial \xi_j}\frac{d\,\text{vol}}{\mathbf{x}}$$

and the ξ's are the Cartesian components of the position vector $\mathbf{x}$.

In fact, (2.6) *is* the form adopted in the present model; it is thus implicitly assumed that any inhomogeneities in the flow do not make a major contribution to the integral appearing in (2.4). The approximation coincides with Chou's proposals. It remains, of course, to prescribe a_{lj}^{mi}. Neither Chou nor Rotta proposed a general form for this fourth-order tensor though both drew attention to the constraints, arising from symmetry and mass conservation, which its components must satisfy, namely

$$a_{lj}^{mi} = a_{lj}^{im} = a_{jl}^{im}, \tag{2.7}$$

$$a_{li}^{mi} = 0. \tag{2.8}$$

Moreover, as Rotta (1951) observed, it follows from Green's theorem that

$$a_{jj}^{mi} = 2\overline{u_m u_i}. \tag{2.9}$$

The form of (2.8) and (2.9) suggested first that a_{lj}^{mi} could be satisfactorily approximated by a *linear* combination of Reynolds stresses involving one (or both) of the velocity components u_m or u_i. Equations (2.7), (2.8) and (2.9) were then sufficient to determine the exact form of the tensor, but the result did not agree with the implications of the data of Champagne, Harris & Corrsin (1970) (to which we shall shortly turn). Accordingly, double products of Reynolds stresses were added, the resultant form satisfying the symmetry requirement of (2.7) being

$$\begin{aligned} a_{lj}^{mi} = {} & \alpha \overline{u_m u_i}\delta_{lj} + \beta(\overline{u_m u_l}\delta_{ij} + \overline{u_m u_j}\delta_{il} + \overline{u_i u_j}\delta_{ml} + \overline{u_i u_l}\delta_{mj}) \\ & + (\gamma\delta_{mi}\delta_{lj} + \eta[\delta_{ml}\delta_{ij} + \delta_{mj}\delta_{il}])\,k + v(\overline{u_m u_j}\,\overline{u_i u_l} + \overline{u_m u_l}\,\overline{u_i u_j})/k \\ & + c_{\phi 2}(\overline{u_m u_i}\,\overline{u_l u_j})/k, \end{aligned} \tag{2.10}$$

where $\alpha, \beta, \gamma, \eta, v$ and $c_{\phi 2}$ are constants. In principle the application of (2.8) and (2.9) enables five of the constants to be determined in terms of the sixth. If, however, v and $c_{\phi 2}$ are to be non-zero, (2.9) cannot be satisfied exactly. We therefore replaced at one point the correlation $\overline{u_m u_j}\,\overline{u_i u_j}$ by $\overline{u_m u_i}k$. With this substitution, the other constants may be expressed as follows in terms of $c_{\phi 2}$:

$$\left.\begin{aligned} &\alpha = (10 - 8c_{\phi 2})/11, \quad \beta = -(2 - 6c_{\phi 2})/11, \\ &\gamma = -(4 - 12c_{\phi 2})/55, \quad \eta = (6 - 18c_{\phi 2})/55, \quad v = -c_{\phi 2}. \end{aligned}\right\} \tag{2.11}$$

The final form of the simulated equations for Reynolds stress may therefore be written as

$$\frac{D\overline{u_i u_j}}{Dt} = -\left[\overline{u_j u_k}\frac{\partial U_i}{\partial x_k} + \overline{u_i u_k}\frac{\partial U_j}{\partial x_k}\right] - \tfrac{2}{3}\delta_{ij}\epsilon - c_{\phi 1}\frac{\epsilon}{k}\left(\overline{u_i u_j} - \delta_{ij}\frac{2k}{3}\right)$$
$$+ (\phi_{ij}+\phi_{ji})_2 + c_s\frac{\partial}{\partial x_k}\frac{k}{\epsilon}\left[\overline{u_i u_l}\frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l}\frac{\partial \overline{u_k u_i}}{\partial x_l} + \overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\right], \quad (2.12)$$

where, for brevity, $(\phi_{ij}+\phi_{ji})_2$ is used to denote the second part of the pressure-strain term simulated by (2.6), (2.10) and (2.11).

Equation (2.12) contains three constants (c_s, $c_{\phi 1}$ and $c_{\phi 2}$), which must be chosen by reference to experimental data. It is convenient to defer selection of the diffusion constant c_s until §4; appropriate values for $c_{\phi 1}$ and $c_{\phi 2}$ will however be indicated now. Uberoi's (1957) measurements of the decay, in the absence of

$c_{\phi 1}$	$c_{\phi 2}$	$(\overline{u_1^2}-\frac{2}{3}k)/2k$	$(\overline{u_2^2}-\frac{2}{3}k)/2k$	$(\overline{u_3^2}-\frac{2}{3}k)/2k$	$-\overline{u_1 u_2}/k$
2·8	0·45	0·135	−0·085	−0·05	0·27
2·5	0·40	0·14	−0·08	−0·06	0·30
Champagne *et al.*		0·14	−0·09	−0·05	0·33

TABLE 1. Reynolds stresses in a homogeneous shear flow

mean strain, of an isotropic field towards isotropy indicate (as Rotta (1962) has noted) that $c_{\phi 1}$ should lie between about 2·6 and 3·0. A value of about 2·5 is suggested by the more recent data of Tucker & Reynolds (1968). To select $c_{\phi 2}$ we consider a homogeneous turbulent field in which U_1 increases linearly with x_2, and U_2 and U_3 are zero. For this flow $\epsilon = -\overline{u_1 u_2}(dU_1/dx_2)$ and hence (2.12) may be reduced to the following simple algebraic forms:

$$(\overline{u_1^2}-\tfrac{2}{3}k)/2k = (4+10c_{\phi 2})/33(c_{\phi 1}-2c_{\phi 2}), \quad (2.13)$$

$$(\overline{u_2^2}-\tfrac{2}{3}k)/2k = (1-14c_{\phi 2})/33(c_{\phi 1}-2c_{\phi 2}), \quad (2.14)$$

$$(\overline{u_3^2}-\tfrac{2}{3}k)/2k = -(5-4c_{\phi 2})/33(c_{\phi 1}-2c_{\phi 2}), \quad (2.15)$$

$$\frac{\overline{u_1 u_2}}{k} = \frac{1}{c_{\phi 1}-2c_{\phi 2}}\left(\frac{6c_{\phi 2}-2}{55} + \frac{2c_{\phi 2}+3}{11}\frac{\overline{u_2^2}}{k} - \frac{6c_{\phi 2}-2}{11}\frac{\overline{u_1^2}}{k}\right)^{\frac{1}{2}}. \quad (2.16)$$

Table 1 compares, for two pairs of values for $c_{\phi 1}$ and $c_{\phi 2}$, the values of the Reynolds stresses given by the above equations with the data of Champagne, Harris & Corrsin (1970). For either pair of constants, agreement between experiment and prediction is seen to be satisfactory, including the non-equality of the normal stresses $\overline{u_2^2}$ and $\overline{u_3^2}$.

3. The rate of dissipation of turbulence energy

For the high Reynolds number flows considered here, the rate of dissipation of turbulence energy ϵ, is equal to $\nu\overline{(\partial u_i/\partial x_l)^2}$. Following Davidov (1961) and Harlow

& Nakayama (1968) an exact transport equation for ϵ may be constructed which for high Reynolds numbers may be expressed as

$$\frac{D\epsilon}{Dt} = -2\nu \frac{\partial U_i}{\partial x_k}\underbrace{\left(\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} + \overline{\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}}\right)}_{\text{(i)}} - 2\nu \underbrace{\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}}}_{\text{(ii)}}$$
$$-2\underbrace{\overline{\left[\nu \frac{\partial^2 u_i}{\partial x_k \partial x_l}\right]^2}}_{\text{(iii)}} - \underbrace{\frac{\partial}{\partial x_k}\overline{u_k \epsilon'}}_{\text{(iv)}} - \frac{\nu}{\rho}\frac{\partial}{\partial x_i}\underbrace{\left[\overline{\frac{\partial p}{\partial x_l}\frac{\partial u_i}{\partial x_l}}\right]}_{\text{(v)}}. \quad (3.1)$$

Our task here (one exactly comparable with that of §2) is to provide reasonable closing approximations of the terms on the right-hand side of (3.1) in terms of $\overline{u_i u_j}$, ϵ and the mean rate of strain. Attention is turned first to the generation term denoted by (i). Since a contraction of the indices yields components of ϵ the following approximation seems appropriate:

$$2\nu\left(\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} + \overline{\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}}\right) = \left(c_{\epsilon 1}\frac{\overline{u_i u_k}}{k} + \tilde{c}_{\epsilon 1}\delta_{ik}\right) \quad (3.2)$$

where $c_{\epsilon 1}$ and $\tilde{c}_{\epsilon 1}$ are constants. In fact, the term containing $\tilde{c}_{\epsilon 1}$ vanishes when (3.2) is multiplied by $\partial U_i/\partial x_k$; thus it need not be considered further.

Rodi (1971) has argued that term (ii), which expresses the generation rate of vorticity fluctuations through the self-stretching action of turbulence, should be considered in conjuction with term (iii), representing the decay of the dissipation rate ultimately through the action of viscosity. Where the Reynolds number is high enough for an inertial subrange to exist, the sum of terms (ii) and (iii) may be taken as being controlled by the dynamics of the energy cascade process transporting energy from low to high wavenumbers and thus as independent of viscosity. For dimensional homogeneity it is concluded that

$$2\left\{\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} + \overline{\left(\nu\frac{\partial^2 u_i}{\partial x_k \partial x_l}\right)^2}\right\} = c_{\epsilon 2}\epsilon^2/k. \quad (3.3)$$

There remain two terms in (3.1) to be considered, both of which express the influence of diffusional transport processes. Term (iv), which accounts for the diffusion of ϵ from velocity fluctuations, is treated in a manner analogous to its counterpart in the stress equations of §2. In appendix A is it shown that a firm pruning of the exact equation for $\overline{\epsilon' u_k}$ leads to the following result:

$$\overline{\epsilon' u_k} = -\frac{c_\epsilon k}{\epsilon}\overline{u_k u_l}\frac{\partial \epsilon}{\partial x_l}. \quad (3.4)$$

Term (v) represents the diffusional transport of ϵ by pressure fluctuations. An exact expression for $\overline{(\partial p/\partial x_l)(\partial u_i/\partial x_l)}$ can readily be constructed which is similar in appearance to (2.4). Like (2.4), the resulting expression contains two groups of terms, one of which arises from an interaction between the mean and turbulent flow field and the other from a self-interaction of the turbulence. Both terms, however, contain higher order derivatives of the mean or fluctuating velocity field than appear in (2.4). It is thus consistent with the level of closure adopted in the equations for Reynolds stress for these pressure diffusion terms to be neglected. The final form of the simulated transport equation for the energy-dissipation rate can thus be expressed as follows:

$$\frac{D\epsilon}{Dt} = -c_{\epsilon 1}\frac{\epsilon\overline{u_i u_k}}{k}\frac{\partial U_i}{\partial x_k} - c_{\epsilon 2}\frac{\epsilon^2}{k} + c_\epsilon \frac{\partial}{\partial x_k}\left(\frac{k}{\epsilon}\overline{u_k u_l}\frac{\partial \epsilon}{\partial x_l}\right). \tag{3.5}$$

The equation contains three constants which must be assigned appropriate values; it is this topic that we consider next.

Measurements of the decay of turbulence behind a grid (e.g. Batchelor & Townsend 1948) have established that at high Reynolds numbers the turbulence energy level varies inversely with distance from the grid ($k = cx_1^{-1}$, say, where c is a dimensional constant particular to the experiment in question). Now for this particular flow, the transport equation for k may exactly be expressed as

$$U_1 dk/dx_1 = -\epsilon$$

and hence $\epsilon = cU_1 x_1^{-2}$. It is readily verified that for (3.5) to be consistent with this decay law $c_{\epsilon 2}$ must equal 2·0.

A further relationship among the constants may be obtained by reference to the properties of the constant-stress layer adjacent to a wall. In such a flow, convective transport is negligible and the production and dissipation rates of turbulence energy are equal, i.e. $\epsilon = -\overline{u_1 u_2}dU_1/dx_2 = u_\tau^2 dU_1/dx_2$. Equation (3.5) may consequently be written as

$$\frac{1}{k}\left(u_\tau^2\frac{dU_1}{dx_2}\right)^2 (c_{\epsilon 1} - c_{\epsilon 2}) + c_\epsilon \frac{d}{dx_2}\left[k\overline{u_2^2}\frac{d^2U_1}{dx_2^2}\bigg/\frac{dU_1}{dx_2}\right] = 0,$$

where x_2 denotes the direction normal to the wall and u_τ the friction velocity. Moreover, on noting the following experimental properties of the flow:

$$dU_1/dx_2 \approx u_\tau/0{\cdot}42x_2, \quad k \approx 3{\cdot}5u_\tau^2, \quad \overline{u_2^2} \approx 1{\cdot}6u_\tau^2,$$

and recalling that $c_{\epsilon 2}$ should take the value 2·0, it follows that

$$c_{\epsilon 1} \approx 2 - 3{\cdot}5c_\epsilon. \tag{3.6}$$

4. A simpler version of the model for boundary-layer flows

In two-dimensional boundary-layer flows (with x_1 the primary direction of flow and x_2 the primary direction of velocity gradient) the shear stress $-\overline{u_1 u_2}$ will usually be the only Reynolds stress to exert significant direct influence on the mean-flow development. Here we exploit this property of boundary-layer flows to deduce a form of turbulence model whose visual appearance (and, equally, the task of incorporating it in a practical calculation procedure) is much simpler than that of the one presented in §§2 and 3 above. There are two basic principles involved; first, that instead of solving transport equations for each of the normal stresses, an equation is provided for their sum (strictly, half their sum), the turbulence energy. Second, the normal-stress terms which remain in the equations for kinetic energy, shear stress and energy-dissipation rate are then taken as proportional to the turbulence energy. The reader may perhaps notice that the latter principle is very similar to one employed by Bradshaw, Atwell & Ferriss (1967). These authors, however, relate the *shear* stress to the turbulence energy and this practice effectively limits their procedure to external wall boundary layers wherein the shear stress does not change sign.

From (2.12) and the defining equations which precede it it may be deduced that the form of the simulated shear stress and turbulence energy equations appropriate to plane boundary layers is

$$\frac{D\overline{u_1u_2}}{Dt} = \left\{(\beta+\alpha-1)\overline{u_2^2}+\beta\overline{u_1^2}+(\eta+\gamma)k-2c_{\phi 2}\frac{(\overline{u_1u_2})^2}{k}\right\}\frac{\partial U_1}{\partial x_2}-c_{\phi 1}\frac{\epsilon}{k}\overline{u_1u_2}$$
$$+c_s\frac{\partial}{\partial x_2}\left\{\frac{k}{\epsilon}\left(\overline{u_1u_2}\frac{\partial\overline{u_2^2}}{\partial x_2}+2\overline{u_2^2}\frac{\partial\overline{u_1u_2}}{\partial x_2}\right)\right\}, \quad (4.1)$$

$$\frac{Dk}{Dt} = -\overline{u_1u_2}\frac{\partial U_1}{\partial x_2}-\epsilon+c_s\frac{\partial}{\partial x_2}\left\{\frac{k}{\epsilon}\left(\overline{u_2^2}\frac{\partial}{\partial x_2}(k+\overline{u_2^2})+\overline{u_1u_2}\frac{\partial\overline{u_1u_2}}{\partial x_2}\right)\right\}. \quad (4.2)$$

Besides ϵ (for which a transport equation is provided below), (4.1) and (4.2) contain $\overline{u_1^2}$ and $\overline{u_2^2}$ as unknowns. It is appropriate to eliminate these by using the plane shear layer results presented in table 1; this is the only *necessary* simplification. It will substantially simplify the final form, however, and, arguably, will not diminish the accuracy of the resultant model if, by the same means, $\overline{u_1u_2}$ also is eliminated in the coefficient of $\partial U_1/\partial x_2$ in (4.1) and in the final term in (4.2). Lastly, the term $\overline{u_1u_2}\partial\overline{u_2^2}/\partial x_2$, which contributes to the shear stress diffusion, is neglected on the grounds that the diffusion term as a whole is mainly of importance where mean velocity gradients are small, and in these regions $\overline{u_1u_2}\partial\overline{u_2^2}/\partial x_2$ will ordinarily be much smaller than $\overline{u_2^2}\partial\overline{u_1u_2}/\partial x_2$.

Thus, with $c_{\phi 1}$ and $c_{\phi 2}$ given the values 2·8 and 0·45, the following pair of equations results:

$$\frac{D\overline{u_1u_2}}{Dt} = -2{\cdot}8\left(\frac{\epsilon}{k}\overline{u_1u_2}+0{\cdot}07k\frac{\partial U_1}{\partial x_2}\right)+c_s\frac{\partial}{\partial x_2}\left\{\frac{k^2}{\epsilon}\frac{\partial\overline{u_1u_2}}{\partial x_2}\right\}, \quad (4.3)$$

$$\frac{Dk}{Dt} = -\overline{u_1u_2}\frac{\partial U_1}{\partial x_2}-\epsilon+0{\cdot}8c_s\frac{\partial}{\partial x_2}\left\{\frac{k^2}{\epsilon}\frac{\partial k}{\partial x_2}\right\}. \quad (4.4)\dagger$$

Moreover, for the subclass of flows now under consideration, the transport equation for the energy dissipation rate, (3.5), becomes

$$\frac{D\epsilon}{Dt} = c_{\epsilon 1}\frac{\overline{u_1u_2}}{k}\frac{\partial U_1}{\partial x_2}-\frac{c_{\epsilon 2}\epsilon^2}{k}+0{\cdot}5c_\epsilon\frac{\partial}{\partial x_2}\left(\frac{k^2}{\epsilon}\frac{\partial\epsilon}{\partial x_2}\right), \quad (4.5)$$

† It is of interest to note that if convective and diffusive transport is neglected (4.3) reduces to:

$$-\overline{u_1u_2} = -0{\cdot}07(k^2/\epsilon)\,\partial U_1/\partial x_2, \quad (4.7)$$

which is equivalent to the formula used to relate the shear stress to the mean rate of strain in many turbulent viscosity models of turbulence (e.g. Spalding 1970; Jones & Launder 1972). Moreover, if $\partial U_1/\partial x_2$ is eliminated from (4.4) by means of (4.6) the energy equation (again with transport neglected) may be manipulated to become $-\overline{u_1u_2}/k = \sqrt{(0{\cdot}07)} = 0{\cdot}27$, which, save for the value of the constant (0·27 instead of 0·3) is identical to Bradshaw, Ferris & Atwell's relation between stress and energy.

where the coefficient of the last term arises from replacing $\overline{u_2^2}$ by $0{\cdot}5k$ (cf. the top line of table 1). Equations (4.3)–(4.5) together with the mean momentum equation

$$\frac{DU_1}{Dt} = -\frac{1}{\rho}\frac{dP}{dx_1} - \frac{\partial}{\partial x_2}\overline{u_1 u_2} \tag{4.6}$$

comprise the proposed form of the model for boundary-layer flows. The model contains six constants to be determined from experiment. Two of these ($c_{\phi 1}$ and $c_{\phi 2}$) have, for algebraic clarity, already been chosen. For the calculations presented in § 5 the remainder have been assigned the values noted in table 2. The entry 'computer optimization' against the diffusion constants c_s and c_ϵ means that for the flows considered in § 5 many calculations were performed in which the constants were systematically varied. The values chosen are those which we believed gave the best overall agreement for the flows considered.

Constant	Values	Basis for choice
$c_{\phi 1}$	2·8	Return to isotropy of distorted turbulence, Rotta (1962)
$c_{\phi 2}$	0·45	Plane homogeneous shear flow, Champagne *et al.* (1970).
c_s	0·08	Computer optimization
$c_{\epsilon 1}$	1·45	Near-wall turbulence (equation (3.6))
$c_{\epsilon 2}$	2·0	Decay of grid turbulence
c_ϵ	0·13	Computer optimization

TABLE 2. The empirical constants

5. Comparison of predictions with experiment

The calculations presented below have all been obtained by solving (4.3)–(4.6) by means of an adapted version of the finite-difference procedure of Patankar & Spalding (1970); details are provided by Hanjalić (1970). The above transport equations are only appropriate for flow regions where the Reynolds number of the turbulence is high, and this means, of course, that in performing the calculations the viscous sublayer region must be excluded. Consequently, for wall boundary layers boundary conditions are applied *near* (rather than *at*) the wall. Thus, except where otherwise stated, the velocity is matched to Patel's (1965) version of the logarithmic law of the wall, the gradient of turbulence energy is set to zero, the mean momentum equation with convection neglected provides a formula for the shear stress and the energy-dissipation rate is equated to the generation rate.

At an axis of symmetry the shear stress is made zero, whereas the *gradient* normal to the flow direction of the other dependent variables is set to zero. Finally, at free boundaries the shear stress is again made zero while the other variables are determined from the degenerate forms of their respective transport equations which result when derivatives with respect to x_2 are set to zero.

The data which were uppermost in our mind when devising the model were our own measurements of fully developed asymmetric flow in a plane channel

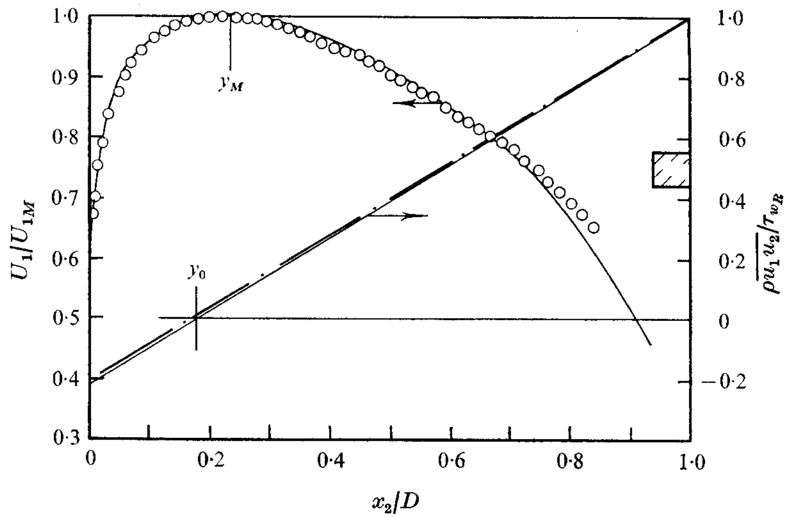


FIGURE 1. Velocity and shear stress in asymmetric channel flow (Hanjalić & Launder 1972). —, predictions, Re_M = 62 000; ○, — - —, experiment, Re_M = 56 600.

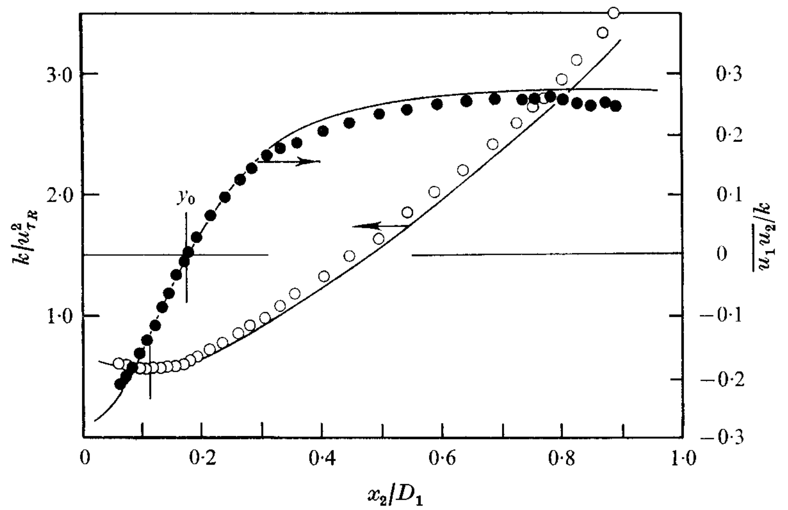


FIGURE 2. Correlation coefficient and turbulence energy in asymmetric channel flow (Hanjalić & Launder 1972). —, predictions, Re_M = 62000; ○, ●, experiment, Re_M = 56000.

(Hanjalić & Launder 1972). The asymmetry had been introduced by fixing ribs of square cross-section aligned normal to the flow and pitched 10 rib heights apart to one wall. Figure 1 shows the calculated and measured distributions of mean velocity and shear stress across the channel for a Reynolds number Re_M (based on maximum velocity U_{1M} and half the distance between plates) of about 60000. The measured shear stress profile is that deduced from the streamwise pressure gradient and a Stanton tube measurement of the smooth-wall stress.

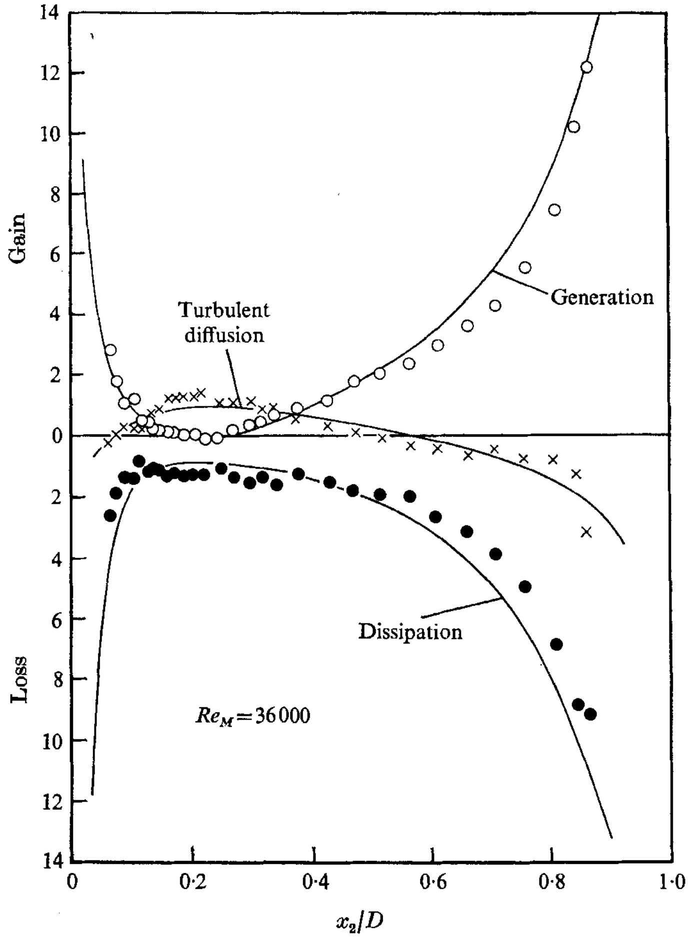


FIGURE 3. Turbulence energy balance in asymmetric channel flow. —, predictions. Experiment: ○, $-(\overline{u_1u_2}\,dU_1/dx_2)D/u_{\tau R}^3$; ×, $-[d\frac{1}{2}(\overline{u_i^2u_2})/dx_2]D/u_{\tau R}^3$; ●, $\epsilon D/u_{\tau R}^3$.

In the calculations, the mean velocity near the roughened wall has been matched to the equation

$$\frac{U_1}{u_{\tau R}} = \frac{1}{0{\cdot}42}\ln\left(\frac{x_2 u_{\tau R}}{\nu}\right) + 3{\cdot}5, \tag{5.1}$$

which had been found by Hanjalić & Launder to correlate their data over an appreciable region near the rough wall. Agreement between calculation and experiment is satisfactory, including the non-coincidence of the positions of maximum velocity and zero shear stress. In figure 2 the hot-wire data of shear stress and turbulence energy are compared with calculated profiles. The strongly asymmetric nature of these profiles is again faithfully reproduced by the predictions. Points to notice include the extensive region on the rough-wall side of the duct where $(\overline{u_1u_2}/k)$ is virtually constant, the absence of any such region near the smooth wall and the non-coincidence of the positions of minimum kinetic energy and zero stress. All these features are well predicted.

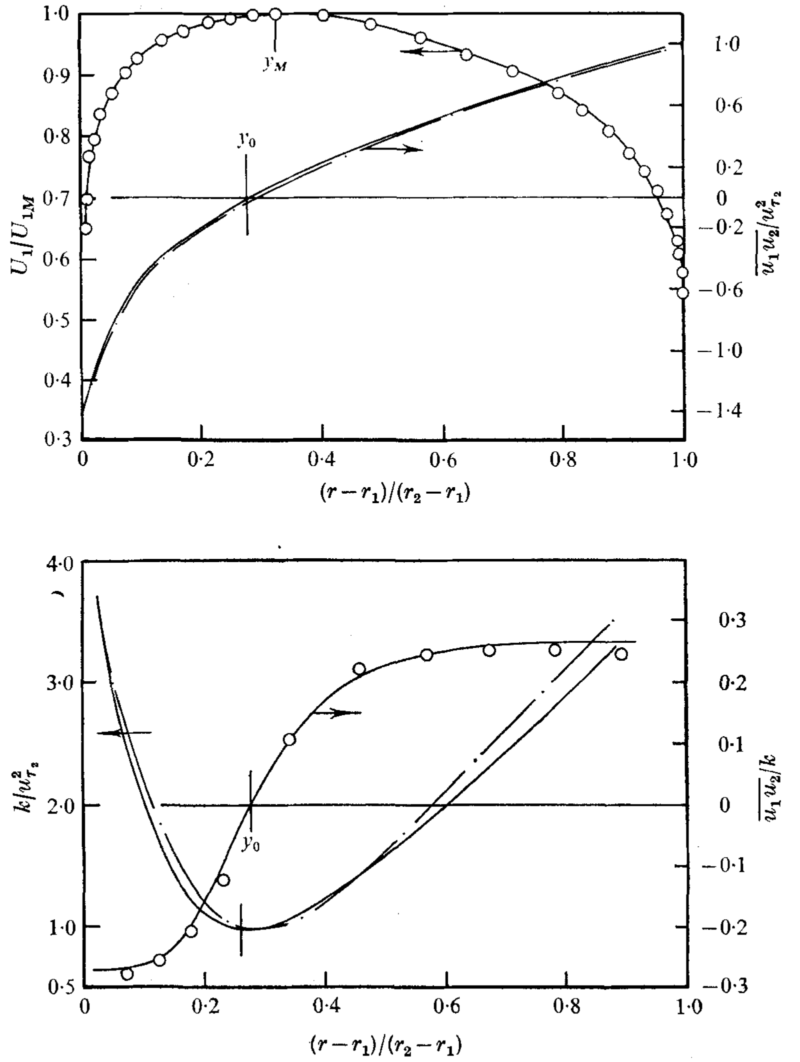


FIGURE 4. Velocity, shear-stress and turbulence energy profiles in symmetric annulus with small radius ratio. $Re = 240\,000$, $r_1/r_2 = 0{\cdot}088$. —, predictions; ○, ●, — - —, experiment (Lawn 1970).

The turbulence energy balance for the above flow is shown in figure 3. The experimental values for generation and diffusion were measured directly; those for dissipation were obtained as the closing term. A feature of this asymmetric channel flow is that the diffusion term is of substantially greater importance than in a smooth channel. The calculated and measured diffusion fluxes are seen generally to be in satisfactory agreement.

We have also made calculations of flow in a plane channel with smooth walls. In this case the predicted profiles of mean velocity and turbulence energy fall between the experimental data of Comte-Bellot (1965) and Laufer (1951) and hence may be said to lie within the experimental uncertainty.

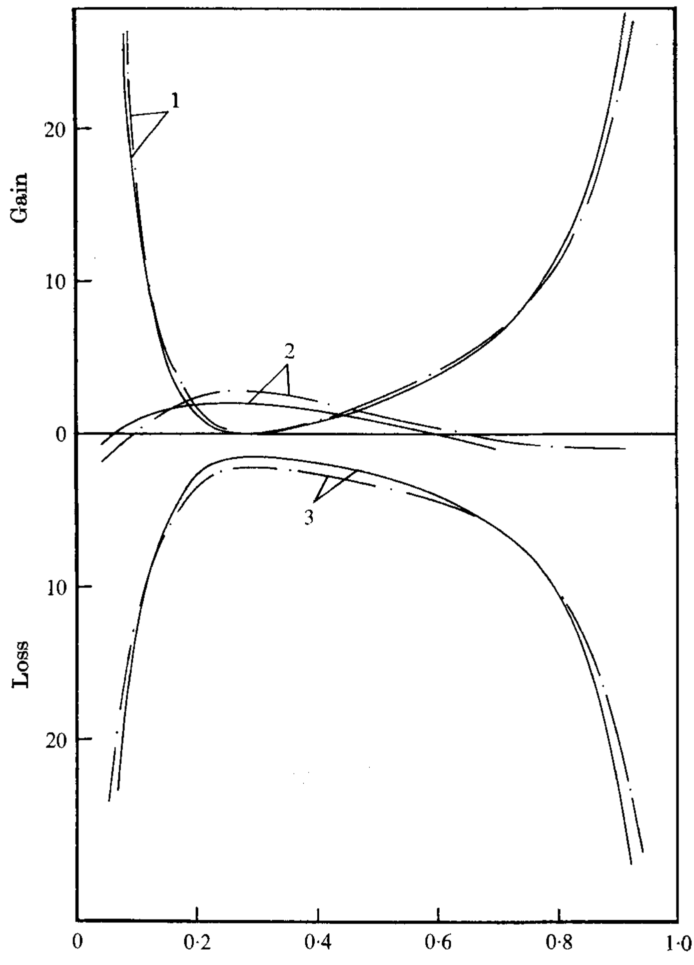


FIGURE 5. Turbulence energy balance in symmetric annulus. $Re = 245000$, $r_1/r_2 = 0{\cdot}088$. —, predictions; — - —, experiment (Lawn 1970); (1), $-[\overline{u_1u_2}(dU_1/dr)](r_2-r_1)/u^3_{\tau_2}$, generation; (2), $-[1/2r(d(\overline{u_i^2u_2}\,r)/dr)](r_1-r_2)/u^3_{\tau_2}$, turbulent diffusion; (3), $\epsilon(r_2-r_1)/u^3_{\tau_2}$, dissipation.

Lawn (1970) has made a very careful study of another strongly asymmetric internal flow: that which arises in a symmetric annulus where the radius of the core tube is only a small fraction of the outer containing tube (0·088). Figure 4 compares his measured profiles of mean velocity, turbulence energy and shear stress with predictions†; for all profiles good agreement is displayed.

† For the boundary condition on mean velocity near the core tube, the additive constant in the logarithmic law was taken as 4·3 (rather than Patel's value of 5·45) in agreement with Lawn's data. A referee has pointed out that, in view of this departure of the velocity from the universal profile, there is some doubt as to whether the boundary condition used for kinetic energy (generation = dissipation) is still appropriate. The comparison of calculations and measurements in figure 5 suggests, however, that this local equilibrium assumption is still adequate.

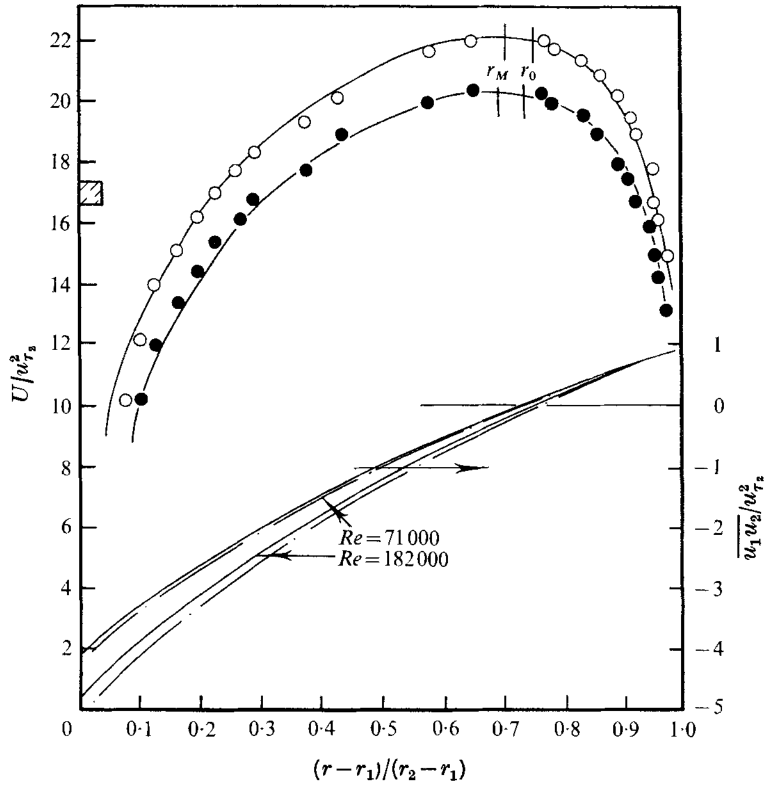
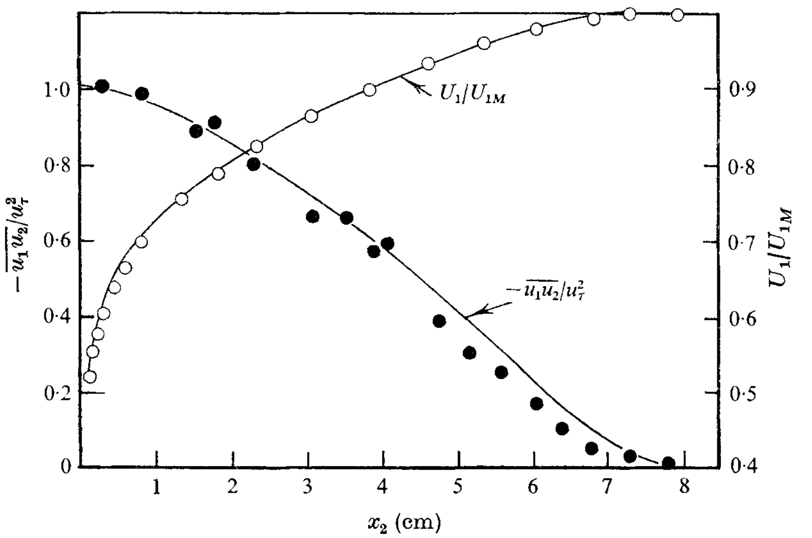


FIGURE 6. Velocity and shear-stress profiles in annulus with rough core. —, predictions. Experiment (Lawn & Hamlin): ●, $Re = 71000$; ○, $Re = 182000$.

An interesting difference between the behaviour here and in the rough–smooth channel emerges on consideration of the relative position of maximum velocity and zero shear stress. In figure 1 it was shown that y_m lies nearer to the surface with the *higher* shear stress than y_0; the reverse is found, however, in the smooth annulus. A turbulence energy balance for the flow is shown in figure 5. Agreement between the profiles of measured and predicted diffusion rates is not quite as close as for the plane channel considered earlier. Considering the difficulty of obtaining measurements of energy diffusion, however, the predictions can be taken as adequate.

The last of the duct flows to be considered is that of Lawn & Hamlin (1969); this is the fully developed flow in a symmetric annulus with a roughened core tube, for a radius ratio of 0·56. The calculated and measured profiles of shear stress and mean velocity are seen, from figure 6, to be in excellent agreement.

Attention is now turned to developing external wall boundary layers. Klebanoff's (1955) measurements of a boundary layer developing in a uniform free stream are compared with predictions in figures 7 and 8; the momentum thickness Reynolds number is 7700. Agreement with the mean velocity and shear stress data is good but, for the energy profile, values are 15–20 % higher than those measured over much of the boundary layer, a discrepancy which is probably



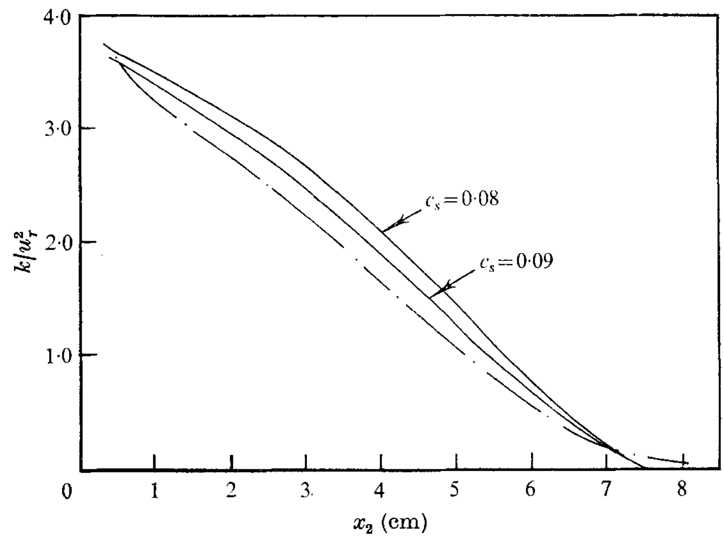


FIGURE 7. Velocity, shear-stress and turbulence energy profiles in the wall boundary layer at uniform pressure. —, predictions. ○, ●, — - —, experiment (Klebanoff).

greater than the uncertainty of the experimental data. Figure 7 shows that the difference can be diminished by increasing the value of c_s; the consequent effects on the velocity and shear stress profiles are not distinguishable on the scale of the figures. Klebanoff obtained his energy balance by measuring the convective transport and production of energy directly, by estimating the dissipation rate from measurements of five of the nine terms in the dissipation term $(\partial u_i/\partial x_j)^2$, and thus obtaining the diffusive flux as the closing term. It is generally supposed, however, that his estimate of ϵ was too low by an appreciable amount and that this led to implausibly large values for the diffusion term; the discrepancy between

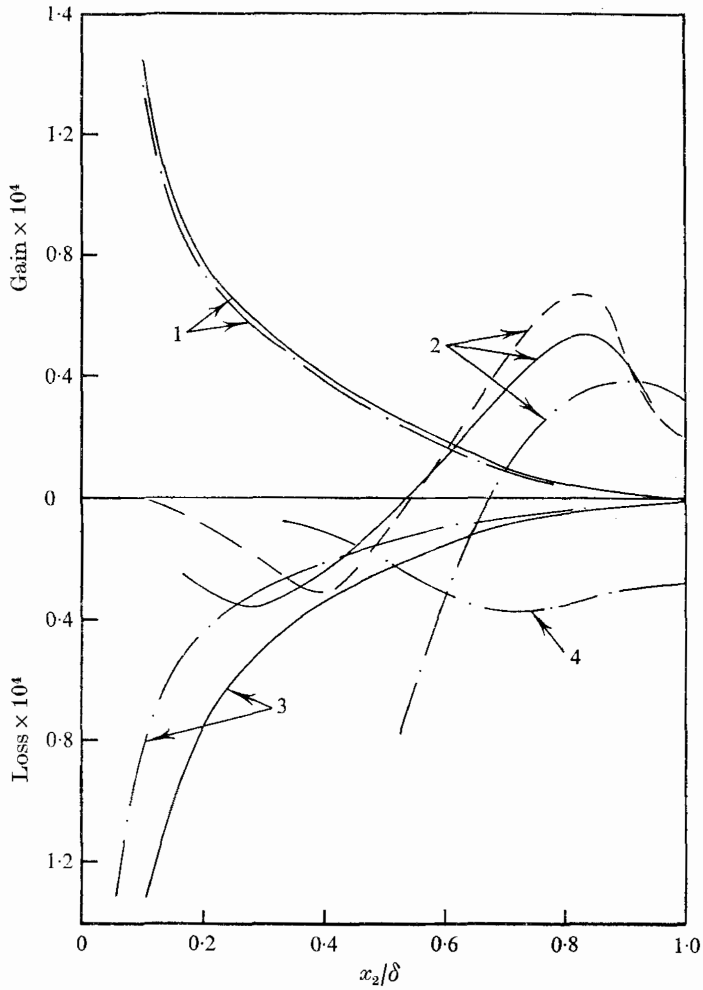


FIGURE 8. Energy balance in wall boundary layer. —, predictions. Experiment: — — —, Klebanoff; — · —, Townsend;

(1), $-\left(\overline{u_1u_2}\dfrac{\partial U_1}{\partial x_2}\right)\dfrac{\delta}{U_{1M}^3}$, generation; (2), $-\left(\dfrac{\partial\frac{1}{2}\overline{(u_i^2u_2)}}{\partial x_2}\right)\dfrac{\delta}{U_{1M}^3}\times 10$, turbulent diffusion;

(3), $\epsilon\dfrac{\delta}{U_{1M}^3}$, dissipation; (4), $\left(U_1\dfrac{\partial k}{\partial x_1}+U_2\dfrac{\partial k}{\partial x_2}\right)\dfrac{\delta}{U_{1M}^3}\times 10$, convection.

measurement and predictions shown in figure 8 is certainly consistent with this supposition. Townsend (1951) has made direct hot-wire measurements of the energy diffusion in a zero-pressure-gradient boundary layer and it is seen from figure 8 that these are in much better agreement with the predicted profile.

Figure 9 compares the predicted velocity and shear-stress profiles in a plane wall jet with the experimental data of Tailland & Mathieu (1967). The shape of the mean velocity profile, like the displacement of the positions of zero stress and maximum velocity, is in agreement with measured results. However, the general level of shear stress is too high, an occurrence which causes the predicted growth

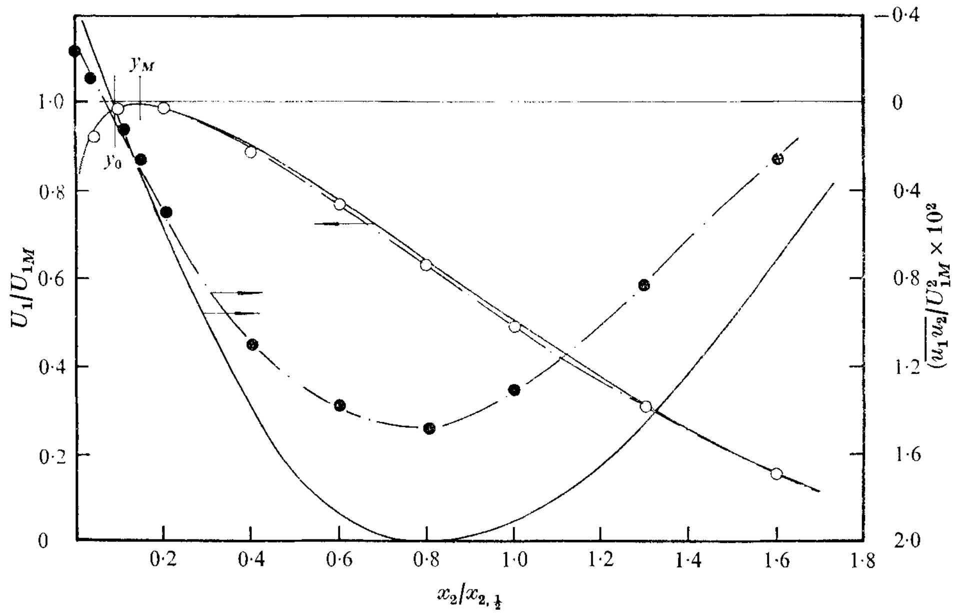


FIGURE 9. Velocity and shear stress profiles in plane wall jet. —, predictions; ○, ●, experiment (Tailland & Mathieu).

rate of the wall jet to be about 20 % greater than measurements suggest. At present we are uncertain whether these differences represent essential deficiencies of the present model or merely indicate that the boundary conditions used are not quite appropriate to the flow in question. For example, as remarked above, we adopted Patel's value for the additive constant in the logarithmic law (5·45). However, in view of the substantial negative gradient of shear stress in the near-wall region of the flow, it is probable that a larger value would have been appropriate. Moreover, there seems at least some doubt as to whether, in the experiments themselves, the free stream was genuinely at rest. Finally, it is remarked that in the calculations the direct influences of the normal stresses have been neglected; the term $-\partial(\overline{u_1^2}-\overline{u_2^2})/\partial x_1$ has been omitted from the mean momentum equation and corresponding terms have been left out of the turbulence transport equations. Although these terms certainly do not dominate the flow's evolution their omission may have contributed to the differences between the measured and the predicted growth rates.

Lastly, we show the outcome of applying the model to the prediction of two free shear flows: the plane jet in stagnant surroundings and the plane mixing layer. The predicted behaviour of the plane jet is compared in figures 10 and 11 with Bradbury's (1965) data. Mean velocity and energy profiles are generally in satisfactory accord but the peak value of the predicted shear stress is some 15 % less than Bradbury's measurements. As a result, the generation term in the energy-balance equation is too small in the neighbourhood of $x_{2,\frac{1}{2}}$, the distance from the axis at which the velocity is one half that on the centre-line. Apart from this defect, the calculated and measured energy balances shown in figure 11 are in satisfactory agreement.

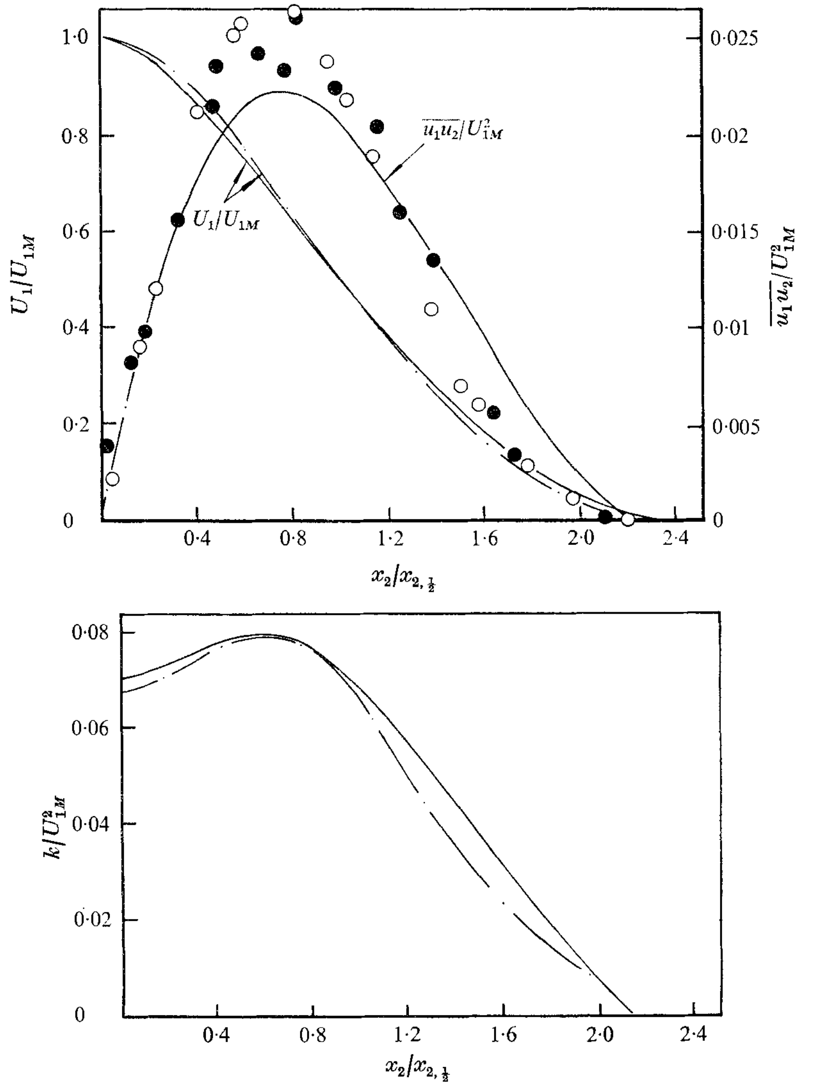


FIGURE 10. Velocity, shear-stress and energy profiles in a plane jet. —, predictions; — · —, ○, ●, experiments at two stations (Bradbury).

The calculations for the plane mixing layer shown in figure 12 likewise display reasonably good agreement with the data of Wygnanski & Fiedler (1970). It is seen, however, that the predictions of the turbulence energy and shear stress are displaced relative to the measurements towards the zero-velocity boundary of the flow, x_{20}; the displacement is about 15 % of the width of the shear flow. A further difference between measurement and prediction is in the rate of spread of the mixing layer. Wygnanski & Fiedler's data show the tangent of the angle of spread to be about 0·20, whereas the calculated value is 0·15. The latter, however, corresponds with the earlier measurements of Liepmann & Laufer (1957).

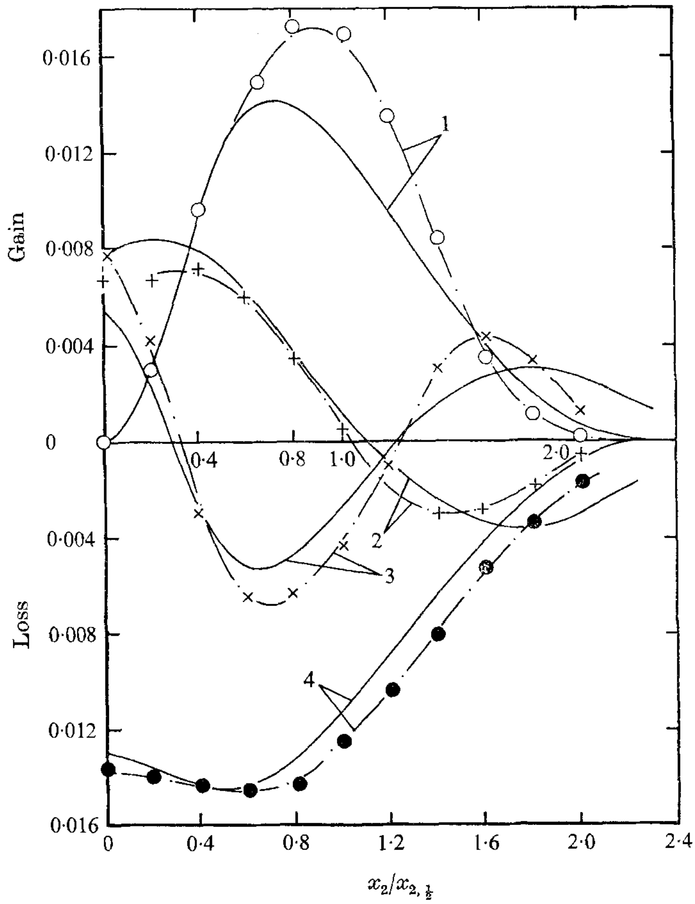


FIGURE 11. Energy balance in a plane jet. —, predictions; — - —, experiment (Bradbury);

$$(1),\ \left(\overline{u_1u_2}\frac{\partial U_1}{\partial x_2}\right)\frac{x_{2,\frac{1}{2}}}{U_{1M}^3};\quad (2),\ \left(U_1\frac{\partial k}{\partial x_1}+U_2\frac{\partial k}{\partial x_2}\right)\frac{x_{2,\frac{1}{2}}}{U_{1M}^3};$$

$$(3),\ \left(\frac{\partial\frac{1}{2}\overline{(u_i^2u_2)}}{\partial x_2}\right)\frac{x_{2,\frac{1}{2}}}{U_{1M}^3};\quad (4),\ \epsilon\frac{x_{2,\frac{1}{2}}}{U_{1M}^3}.$$

6. Concluding remarks

The preceding paragraphs have drawn comparisons between experimental data of various quasi-parallel shear flows and computer solutions of the same flow based upon the model of §4. In many contexts, the range of flows examined would be considered a wide one and the agreement between prediction and measurement good.† We should remember, however, that these flows are still appreciably less

† In addition to the flows considered in §4, the reader is reminded that the more general form of the model presented in §§2 and 3 was also consistent with experimental data on the decay of turbulence behind a grid, the Reynolds stress levels in a plane homogeneous shear layer and the return (in the absence of mean strain) of distorted turbulence towards isotropy.

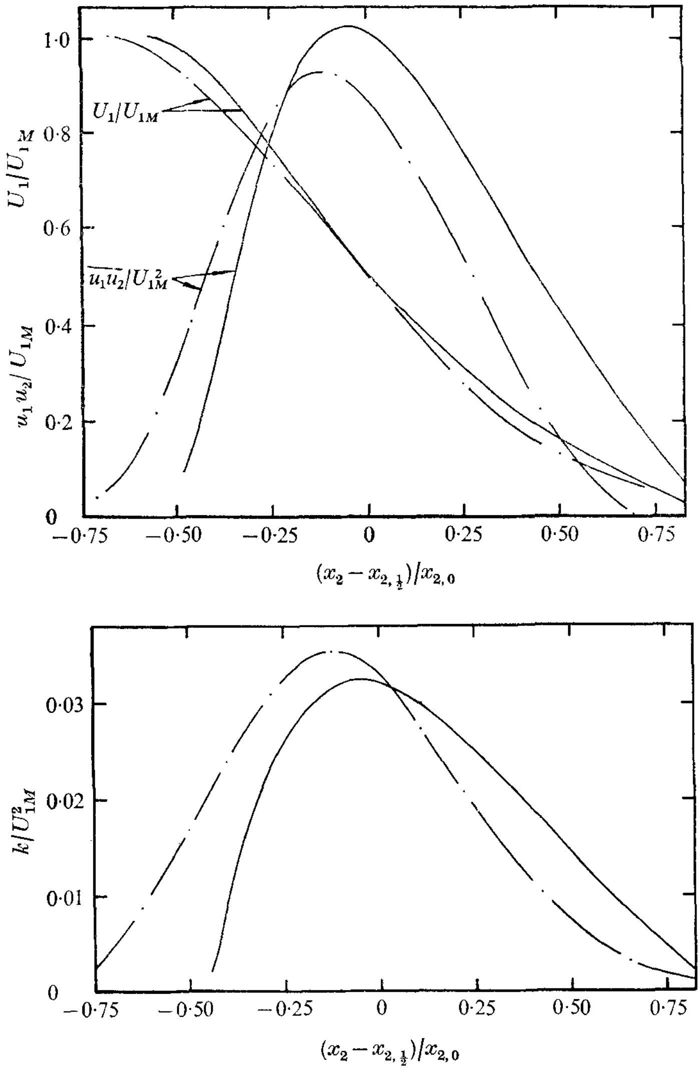


FIGURE 12. Velocity, shear-stress and energy profiles in plane mixing layer. —, predictions; — - —, experimental (Wygnanski & Fielder 1970).

complex than are the majority of industrially important shear flows for which calculation procedures are needed. For such flows we should need to employ the complete stress model rather than the simplified version of §4. In conclusion therefore, it is perhaps appropriate that the approximations made in §§2 and 3 to procure closure should be rescrutinized.

The simulated terms in the Reynolds stress equations lend themselves to more direct comparison with experiment than do those in the dissipation equation and, for this reason, they are considered first. The comparisons made with the measurements of kinetic energy diffusion in figures 3, 5, 8 and 11 provide substantial support for the gradient-diffusion hypothesis represented by equation (2.3). The outcome is not altogether surprising for, as is shown in appendix A, the representation follows from the neglect of convective transport of the triple correlation.† Whether such an approximation of diffusive transport will prove adequate in flows with recirculation remains to be seen. The least controversial of the approximations concerns the representation of the dissipation terms in the stress equations, for at high enough Reynolds numbers the dissipative motions will assuredly be isotropic. Of course, in flows past solid boundaries there must inevitably be a region near the surface where the Reynolds number of the turbulence is low and here notions of local isotropy must be discarded. In these regions, not withstanding some exploratory calculations by Donaldson (1968), there remains considerable doubt as to the appropriate modelling of the dissipation processes (and of the other terms in the stress equations also). We regard this as an important area for further research.

The experimental evidence in support of Rotta's approximation of the pressure strain term has been cited in § 2 and by several other workers. Suffice it to add here that it combines the virtues of physical plausibility and algebraic simplicity. The latter cannot be claimed for our simulation of the second part of the pressure strain term, but by adopting a relatively general form for the tensor a_{lj}^{mi} we were able to satisfy a great many kinematic constraints and this has apparently led to a satisfactory approximation of the term.‡ We believe that near a wall, where the rate of change of mean velocity gradient is large, it will be desirable to improve on the assumption that $\partial U_l/\partial x_m$ is uniform in the integral of (2.4), at any rate if the normal stresses are to be well predicted. However when the flow is nearly parallel to the wall the normal stresses do not exert much direct influence on the mean flow development so the matter may be only of academic interest.

Turning now to the equation for dissipation, it may be said that the form chosen for both the generation and destruction terms is suggested by arguments of local isotropy and is consistent with the practice adopted in closing the stress equations. The forms adopted for these terms are, in fact, equivalent to the approximations of Chou (1945), Davidov (1961) and Daly & Harlow (1970). In treating the diffusion of dissipation Davidov provided a closed set of transport equations for each component $\overline{\epsilon' u_k}$; we decided not to adopt such an elaborate closure. As is shown in appendix A, simplification of those equations leads to the much more tractable gradient-like representation adopted in § 3. If this form should prove to possess too limited validity, the more general version, equation (A 7), could be employed without significantly increasing computation time.

Lastly we draw attention to the absence from our model of any explicit recognition of the intermittent character of the turbulence near the boundary of the shear flow with a quiescent stream. Of course, for the engineer, the more

† Of the flows where diffusion measurements were available to us, two were fully developed flows where convection is, by definition, zero and the other two were equilibrium flows in zero pressure gradients, where convective transport is small.

‡ The second part of the pressure–strain correlation is instrumental in causing turbulence-driven secondary flows in straight ducts. Productions of this type of flow thus provide a very sensitive indicator of the satisfactoriness (or otherwise) of the approximation chosen for that term. It appears from the calculations of Launder & Ying (1971) that the form given in § 2 does predict this occurrence and the effect of turbulence driven secondary flows in square ducts within the accuracy of present measurements.

of the details of turbulence that can be ignored the better, but it is perhaps significant that our calculations are in less satisfactory agreement with measurements in those flows where a free-stream boundary is present, particularly where the stream is at rest. Whether, in practice, the aerodynamicist will ever find it economically worth while to calculate the time-dependent behaviour of the undulating outer edge of the shear flow seems, however, to be very much a matter of conjecture.

The research whose outcome is presented in this paper has been sponsored by the Berkeley Nuclear Laboratories of the CEGB. We are pleased to acknowledge this support and also the keen interest shown in the work by the Board's staff. We also wish to record that the work has benefited from conservations with colleagues at Imperial College, particularly with Mr P. Bradshaw, Dr W. P. Jones and Mr W. Rodi.

Appendix A. Simulation of the turbulent diffusive transport

Some support for the closure approximations expressed by (2.3) and (3.4) may be drawn from consideration of the exact transport equations for the correlation in question. Let us first consider the equation for triple velocity correlation, which for high turbulence Reynolds numbers may be written as

$$\begin{aligned}\frac{D\overline{u_i u_j u_k}}{Dt} = &-\left\{\overline{u_i u_j u_l}\frac{\partial U_k}{\partial x_l}+\overline{u_j u_k u_l}\frac{\partial U_i}{\partial x_l}+\overline{u_k u_i u_l}\frac{\partial U_j}{\partial x_l}\right\} &\text{I}\\ &+\left\{\overline{u_i u_j}\frac{\partial \overline{u_k u_l}}{\partial x_l}+\overline{u_j u_k}\frac{\partial \overline{u_i u_l}}{\partial x_l}+\overline{u_k u_i}\frac{\partial \overline{u_j u_l}}{\partial x_l}\right\} &\text{II}\\ &-\frac{\partial}{\partial x_l}\overline{u_i u_j u_k u_l} &\text{III}\\ &-\left\{\overline{u_i u_k\frac{\partial p}{\partial x_k}}+\overline{u_j u_k\frac{\partial p}{\partial x_i}}+\overline{u_k u_i\frac{\partial p}{\partial x_j}}\right\}. &\text{IV}\end{aligned} \tag{A 1}$$

We simplify the quadruple correlation III by recourse to a proposition of Millionshtchikov (1941), i.e. that when the triple correlations are small and their distribution properties do not differ substantially from those of a Gaussian one, the quadruple correlations may be approximated in terms of the second-order correlations by the formulae which are strictly fulfilled for the normal law. Thus it is presumed that

$$\overline{u_i u_j u_k u_l} = \overline{u_i u_j}\,.\,\overline{u_k u_l}+\overline{u_i u_k}\,.\,\overline{u_j u_l}+\overline{u_i u_l}\,.\,\overline{u_k u_j} \tag{A 2}$$

and, hence, that the sum of terms II and III may be written as

$$\text{II}+\text{III} = -\left\{\overline{u_i u_l}\frac{\partial \overline{u_j u_k}}{\partial x_l}+\overline{u_j u_l}\frac{\partial \overline{u_k u_i}}{\partial x_l}+\overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\right\}. \tag{A 3}$$

From an examination of Hanjalić & Launder's (1972) measurements of asymmetric flow in a plane channel we have found that, in their experiments, the right-hand side of (A 3) was generally of the same sign as and several times larger than term I in (A 1), especially in regions where the triple velocity correlations were of importance. For wall boundary layers, at any rate, it is thus reasonable that term I should be neglected.

There remains the correlation between pressure fluctuations and Reynolds stress (IV) to be considered. An exact expression for this term may be arrived at by precisely the same path as that leading to equation (2.4). As Chou (1945) has shown, the resultant expression contains two types of term, one involving just fluctuating velocity correlations and the other expressing the influence of the mean rate of strain on the pressure-stress correlation. It is consistent with our practice in arriving at (2.6) and (2.10) to approximate the latter process by groups of terms of the form $\overline{u_p u_q u_r}\,\partial U_l/\partial x_m$. We have, however, neglected correlations of this type in term I and we likewise do so here.† Accordingly the pressure-stress correlation is assumed to be adequately represented by

$$-\left\{\overline{u_i u_j \frac{\partial p}{\partial x_k}}+\overline{u_j u_k \frac{\partial p}{\partial x_i}}+\overline{u_k u_i \frac{\partial p}{\partial x_j}}\right\} \propto -\frac{\epsilon}{k}\,\overline{u_i u_j u_k}, \tag{A 4}$$

the form of the right-hand side being suggested by the corresponding approximation in the Reynolds stress equation (equation (2.5)).

Finally, on neglect of the convective transport term from (A 1) and with the substitution of the approximations discussed in the above paragraph, the following algebraic expression emerges for $\overline{u_i u_j u_k}$:

$$\overline{u_i u_j u_k} = -c_s \frac{k}{\epsilon}\left\{\overline{u_i u_l}\frac{\partial \overline{u_j u_k}}{\partial x_l}+\overline{u_j u_l}\frac{\partial \overline{u_k u_i}}{\partial x_l}+\overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\right\}, \tag{A 5}$$

which is identical to (2.3) in the main text.

A similar treatment may likewise be applied to the components of the diffusion of dissipation, $\nu\overline{u_k(\partial u_i/\partial x_l)^2}$, which for brevity we denote by Q_k. By applying to the exact transport equation for Q_k approximations at the same level as those used in §3 to close the equation for ϵ, the following form is obtained:

$$\frac{DQ_k}{Dt} = -Q_l\frac{\partial U_k}{\partial x_l}-\overline{u_k u_l}\frac{\partial \epsilon}{\partial x_l}-c_{Q1}\epsilon\frac{\partial \overline{u_k u_l}}{\partial x_l}-c_{Q2}\frac{\epsilon}{k}Q_k. \tag{A 6}$$

On neglecting the convective transport, (A 6) may be rearranged to give the following explicit algebraic equation for Q_k:

$$Q_k \propto -\frac{k}{\epsilon}\left[\overline{u_k u_l}\frac{\partial \epsilon}{\partial x_l}+Q_l\frac{\partial U_k}{\partial x_l}+c_{Q1}\epsilon\frac{\partial \overline{u_k u_l}}{\partial x_l}\right]. \tag{A 7}$$

Now, Q_k is mainly of importance in the vicinity of a wall where gradients in dissipation rates are large. It is thus safe to assume that the last term in square brackets is small in comparison with the first. Moreover, in boundary-layer flows, at any rate, the second term in brackets is negligible too, since the subscript k will denote the direction normal to the main flow, and U_k and its derivatives are negligibly small. Under these circumstances the equation is adequately approximated by

$$Q_k \equiv \overline{u_k \epsilon'} = -c_\epsilon \frac{k}{\epsilon}\overline{u_k u_l}\frac{\partial \epsilon}{\partial x_l}, \tag{A 8}$$

which is the form adopted in §3.

† Though admittedly these pressure-stress terms are not *necessarily* small even if term I is.

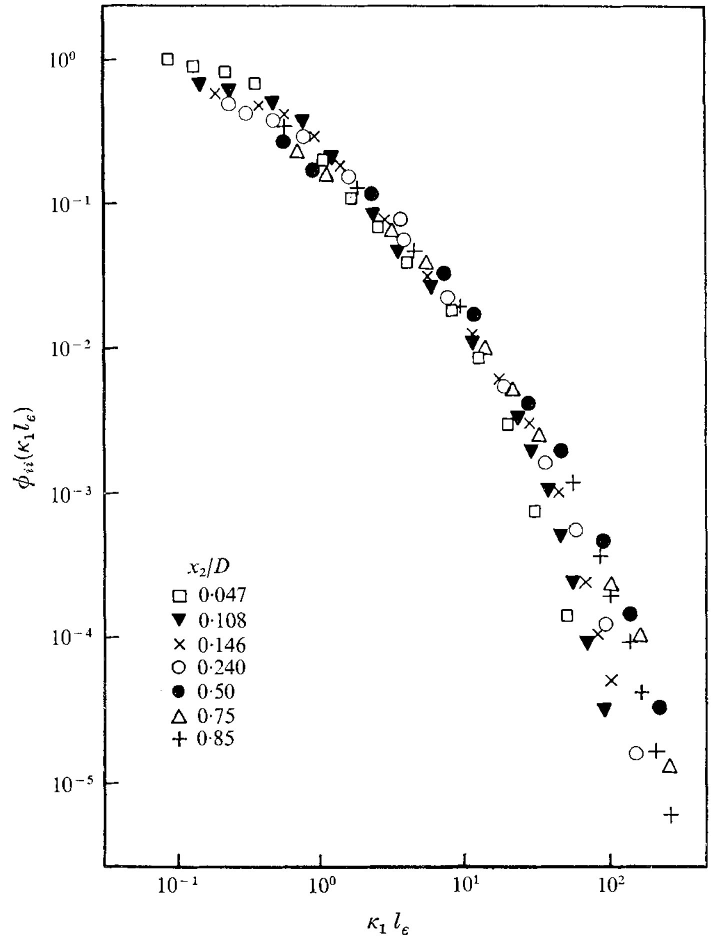


FIGURE 13. Energy spectra in asymmetric channel.

Appendix B. The dissipation length as normalizing scale for one-dimensional spectra

The analysis of the main text brings into prominence the scalar turbulence quantities k and ϵ. Indeed, for boundary-layer flows it is these variables (together with the shear stress $-\overline{u_1 u_2}$) which are determined by way of transport equations. Implicit in the evolution of the forms of these equations is the notion that the directly influential parts of the turbulent motion can be characterized by a single length-scale parameter. The scale which suggests itself is the so-called dissipation length scale l_ϵ defined as $k^{\frac{3}{2}}/\epsilon$.

To provide some basis for assessing the reasonableness of this assumption, figures 13, 14 and 15 show some of our measurements (Hanjalić & Launder 1972) of one-dimensional spectra normalized with the dissipation length scale. The flow geometry, as mentioned in the main text, is a plane channel with one rough and one smooth surface. This gives rise to two quite dissimilar flows in the vicinity of

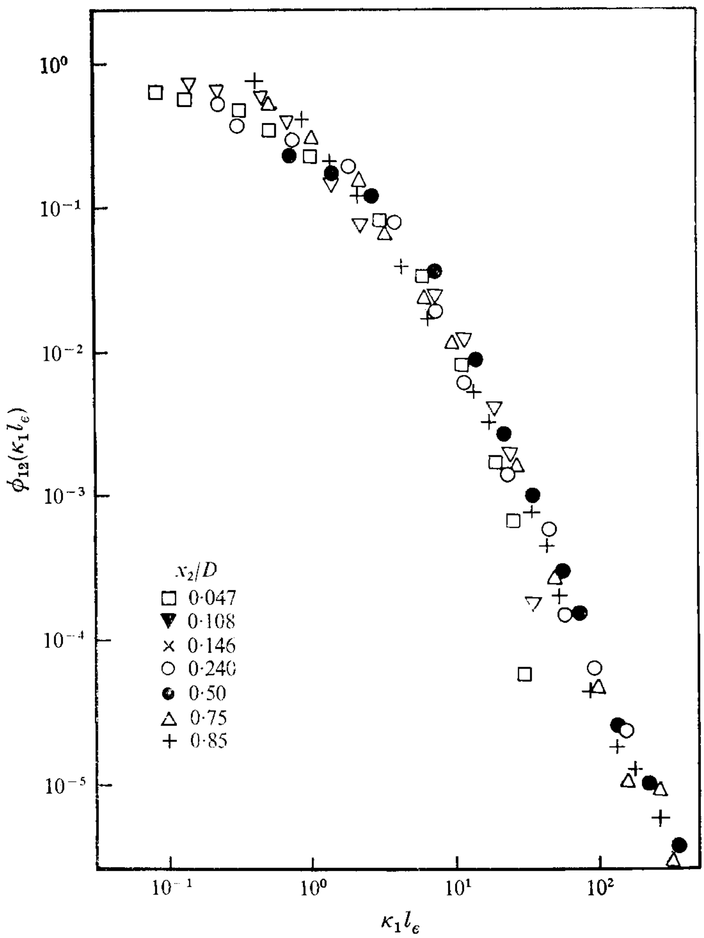


FIGURE 14. Shear-stress spectra in asymmetric channel.

the walls dominated by the adjacent surface and, over a central region of the channel, to a complex flow structure wherein the two wall flows interact. The flow considered is thus not a simple one and it arguably provides a substantial test of the single-length-scale hypothesis.

Figure 13 shows the normalized energy spectra $\phi_{ii}(\kappa_1 l_\epsilon)$† at seven positions in the channel from near the smooth surface ($x_2/D = 0$) to near the rough wall ($x_2/D = 1$). Clearly for values of $\kappa_1 l_\epsilon$ up to about 20 the curves are sensibly universal. As is seen in figure 14, the shear-stress spectrum displays a roughly comparable level of universality over the same range of wavenumbers. Here it is

† The normalized spectral density $\phi_{ij}(\kappa_1 l_\epsilon)$ is defined as

$$\phi_{ij}(\kappa_1 l_\epsilon) \equiv F_{ij}(\kappa_1 l_\epsilon)/\overline{u_i u_j},$$

where F_{ij} is the un-normalized spectra of $\overline{u_i u_j}$ $\left(\text{thus } \int_0^\infty F_{ij}(\kappa_1 l_\epsilon)\, d(\kappa_1 l_\epsilon) = \overline{u_i u_j}\right)$. Correspondingly, for the spectra of triple correlation

$$\phi_{ij,l}(\kappa_1 l_\epsilon) \equiv \overline{u_i u_j(\kappa_1 l_\epsilon) u_l(\kappa_1 l_\epsilon)}/\overline{u_i u_j u_l}.$$

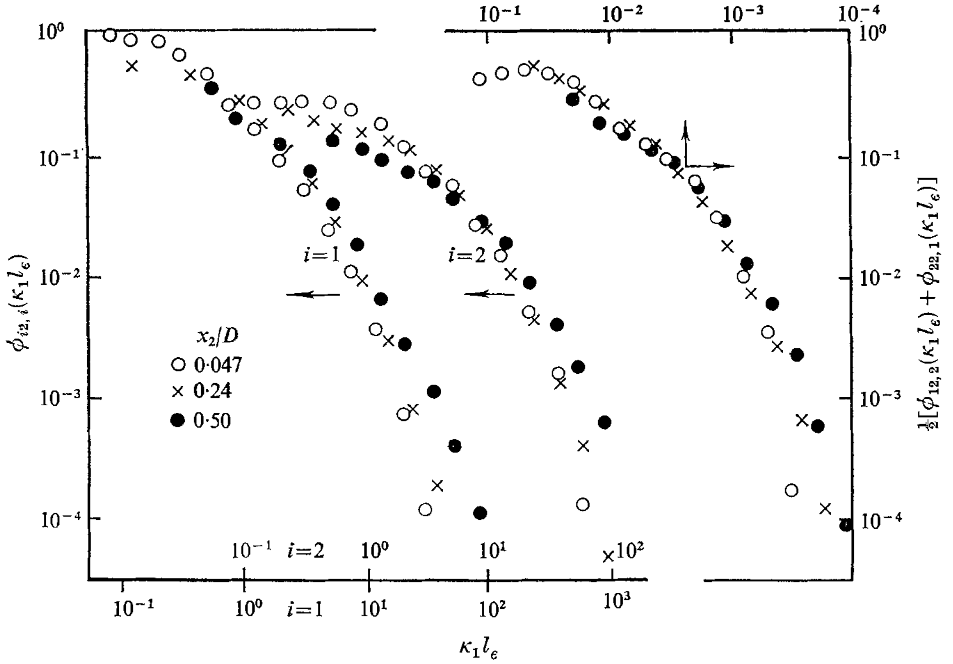


FIGURE 15. Triple correlation spectra in asymmetric channel.

mentioned, however, that the spectrum measured at $x_2/D = 0{\cdot}146$ (which is approximately coincident with the surface of zero shear stress) has not been included. It was left out because $\phi_{12}(\kappa_1 l_\epsilon)$ took values of *opposite sign* in different parts of the spectrum. (The result is attributable to a substantial diffusion of stress from the rough wall region in a middle range of wavenumbers.) So, while overall the spectrum is reasonably universal, its profile near the plane of zero shear stress is quite different from those at other locations in the flow.

Figure 15 shows normalized spectra of triple correlations representing the diffusion flux of shear stress ($\frac{1}{2}[\phi_{12,2}(\kappa_1 l_\epsilon)+\phi_{22.1}(\kappa_1 l_\epsilon)]$) and of components of the normal stresses. For these measurements only three positions in the channel were examined. Here there is more variation from one position to another than was the case with the double correlation spectra. Bearing in mind, however, the difficulty of obtaining accurate spectra of triple correlations, the scatter may be considered to be not unreasonably large.

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